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A LITTLEWOOD-RICHARDSON RULE FOR GRASSMANNIAN PERMUTATIONS

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Abstract. We give a combinatorial rule for computing intersection numbers on a flag manifold which come from products of Schubert classes pulled back from Grassmannian projections. This rule generalizes the known rule for Grassmannians.

Introduction

One of the main open problems in Schubert calculus is to find an analog of the Littlewood-Richardson rule for flag manifolds [Sta00, Problem 11], and more generally to find combinatorial formulae for intersection numbers of Schubert varieties. This problem was recently solved by Coskun for two-step flag manifolds [Co07].

We give such a combinatorial interpretation for intersection numbers of Grassmannian Schubert problems on any type A flag manifold. This number counts certain objects that we call filtered tableaux which satisfy conditions coming from the Schubert problem. When the flag manifold is a Grassmannian this coincides with a standard interpretation of these numbers obtained from the Littlewood-Richardson rule. Grassmannian Schubert problems on the flag manifold were studied in [RSSS06]; they are exactly the Schubert problems which appear in the generalization of the Shapiro conjecture to flag manifolds given there.

In Section 1 we define filtered tableaux, give an example, and state our formula, which we prove in Section 2. Our proof uses some identities of [BS98] which were established using geometry, and is thus not completely combinatorial. In Section 3 we explain how our formula relates to one coming from Monk's formula [Mon59] and discuss how to give a purely combinatorial proof based on the rule of Kogan [Kog01].

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1. A Littlewood-Richardson rule for Grassmannian Schubert problems

For background on flag manifolds and Schubert calculus, see [Ful97]. We fix a positive integer n throughout. Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a non-empty subset of $[n-1] :=$

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$\{1, 2, \ldots, n{-}1\}$, which we write in increasing order

$$\alpha \,:\; 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m < \alpha_{m+1} = n\,.$$

A *partial flag of type α* is a sequence $F_\bullet$ of linear subspaces in $\mathbb{C}^n$

$$F_\bullet \,:\; \{0\} \;\subset\; F_1 \;\subset\; F_2 \;\subset\; \cdots \;\subset\; F_m \;\subset\; \mathbb{C}^n\,,$$

where $\dim F_i = \alpha_i$. The set $\mathcal{F}\ell_\alpha$ of all flags of type α is a complex manifold of dimension

$$\dim(\alpha) \;:=\; \sum_{i=1}^{m} (n - \alpha_i)(\alpha_i - \alpha_{i-1})\,.$$

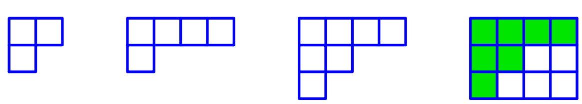
Schubert varieties and classes in $\mathcal{F}\ell_\alpha$ are indexed by permutations w of $\{1, 2, \ldots, n\}$ whose descent set is contained in α. For a permutation w, let σ_w be the class of the Schubert variety corresponding to w, following the conventions in [Ful97]. Its cohomological degree is $2\ell(w)$, where $\ell(w)$ counts the number of inversions $\{i < j \mid w(i) > w(j)\}$ of w.

If $\beta \subset \alpha$ is another subset then there is a projection $\pi_{\alpha,\beta}\colon \mathcal{F}\ell_\alpha \to \mathcal{F}\ell_\beta$ whose fibres are products of flag varieties. When $\beta = \{b\}$ is a singleton, $\mathcal{F}\ell_\beta$ is the Grassmannian $\mathrm{Gr}(b, n)$ of b-planes in $\mathbb{C}^n$. In this case, we write π_b for $\pi_{\alpha,\beta}$. We note that $\pi^*_{\alpha,\beta}\sigma_w$ is just the Schubert class $\sigma_w \in H^*(\mathcal{F}\ell_\beta)$.

Schubert classes in $\mathrm{Gr}(b, n)$ are also indexed by partitions λ, which are northwest-justified arrays of boxes in a $b \times (n - b)$ rectangle, $\square_b$. Associated to a partition λ is the *Grassmannian permutation* w with *shape* λ and descent at b. This permutation has a unique descent at b, and its first b values are

$$w(i) \;=\; i + \lambda(b + 1 - i) \qquad \text{for} \quad i = 1, \ldots, b\,.$$

Here, $\lambda(i)$ denotes the number of boxes in row i of λ. We write σ_λ for the Grassmannian Schubert class σ_w. Here are three partitions with $b = 3$ and $n = 7$; the third is also drawn inside $\square_3$. They correspond to the Grassmannian permutations 1352467, 1372456, and 2471356.



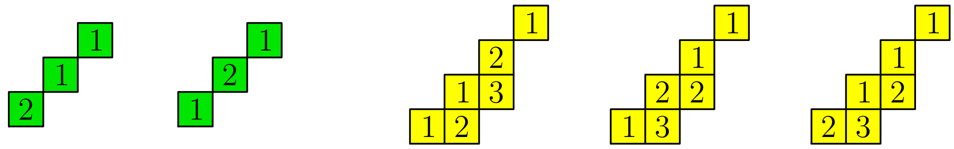
Let $|\lambda|$ be the number of boxes in λ. This is half the cohomological degree of the Schubert class σ_λ and is the complex codimension of the associated Schubert variety.

The Littlewood-Richardson rule for the Grassmannian expresses a product $\sigma_\lambda \cdot \sigma_\mu$ of two Schubert classes as a sum of classes σ_ν where $\lambda, \mu \subset \nu$ with $|\nu| = |\mu| + |\lambda|$. In this rule, the coefficient $c^{\nu/\mu}_\lambda$ of σ_ν is the number of *Littlewood-Richardson tableaux* of skew shape $\nu/\mu := \nu - \mu$ and content λ. These are fillings of the boxes in ν/μ with positive integers such that

(i) The entries weakly increase left-to-right across each row and strictly increase down each column.
(ii) The number of js in the filling is equal to $\lambda(j)$, the number of boxes in row j of λ.
(iii) If we read the entries right-to-left across each row and from the top row to the bottom row, then at every step we will have encountered at least as many occurrences of i as of $i{+}1$ for each positive integer i.

For example, here are some Littlewood-Richardson tableaux.

(1.1)



A *Grassmannian Schubert class* in the cohomology ring of $\mathcal{F}\ell_\alpha$ is the pullback of a Schubert class along a projection to a Grassmannian. That is, it has the form $\pi_b^*\sigma_\lambda$ where $b \in \alpha$ and $\lambda \subset \square_b$. These are indexed by pairs (b, λ) with $\lambda \subset \square_b$.

A *Grassmannian Schubert problem* is a list $((a_1, \lambda_1), \ldots, (a_s, \lambda_s))$ with $a_1 \leq \cdots \leq a_s$. We require that for every $i = 1, \ldots, s$ we have $a_i \in \alpha$ and $\lambda_i \subset \square_{a_i}$, and also

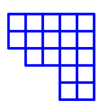
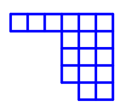
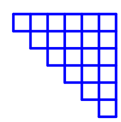
$$|\lambda_1| + |\lambda_2| + \cdots + |\lambda_s| \;=\; \dim(\alpha)\,. \tag{1.2}$$

By the dimension condition (1.2), we have

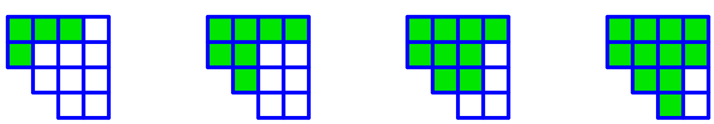
$$\prod_{i=1}^{s} \pi_{a_i}^* \sigma_{\lambda_i} \;\in\; H^{2\dim(\alpha)}(\mathcal{F}\ell_\alpha) \;=\; \mathbb{Z} \cdot [\mathrm{pt}]_\alpha\,,$$

where $[\mathrm{pt}]_\alpha$ is the class of a point in $\mathcal{F}\ell_\alpha$. The problem that we solve is to give a combinatorial formula for the coefficient of $[\mathrm{pt}]_\alpha$ in this product. Note that if $\alpha \supsetneq \{a_1, \ldots, a_s\}$ this coefficient is zero (e.g. by [Knu00, Lemma 1]), and so we will generally assume that $\alpha = \{a_1, \ldots, a_s\}$.

Write ∇_α for the union of all rectangles $\square_a$ for each $a \in \alpha$, where the rectangles all share the same upper right corner. Here are three such shapes when $n = 7$.

$\nabla_{235} =$  $\nabla_{145} =$  $\nabla_{[6]} =$ 

A *shape* $\mu \subset \nabla_\alpha$ is a subset of boxes which are northwest justified. For example, when $n = 6$, the shaded boxes are four shapes in ∇_{234}.



Definition 1.1. Let $\Lambda = ((a_1, \lambda_1), \ldots, (a_s, \lambda_s))$ be a Grassmannian Schubert problem. Set $\alpha = \{a_1, a_2, \ldots, a_s\}$ and fix a shape $\mu \subset \nabla_\alpha$. A *filtered tableau $T_\bullet$ with shape μ and content Λ* is a sequence

$$\mu_\bullet \;:\; \emptyset = \mu_0 \;\subset\; \mu_1 \;\subset\; \mu_2 \;\subset\; \cdots \;\subset\; \mu_{s+1} \;\subset\; \mu_s = \mu$$

of shapes together with fillings $T_1, \ldots, T_s$ of the skew shapes μ_i/μ_{i-1} by positive integers which satisfy the following properties.

(1) The skew shape μ_i/μ_{i-1} must fit entirely within the rectangle $\square_{a_i} \subset \nabla_\alpha$.
(2) The filling T_i is a Littlewood-Richardson tableau of content λ_i.

Note that we must have $|\mu| = |\lambda_1| + \cdots + |\lambda_s|$.

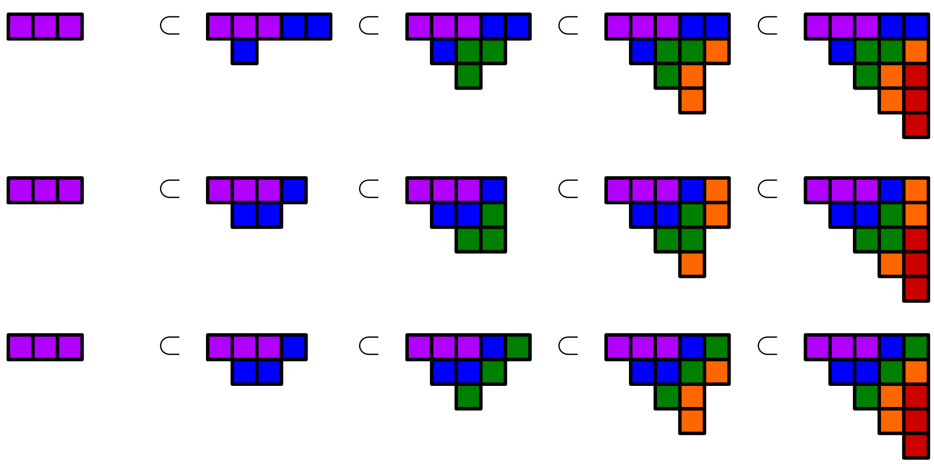
An induction shows that the coefficient of $[\mathrm{pt}]_b = \sigma_{\square_b}$ in a product $\sigma_{\lambda_1}\cdots\sigma_{\lambda_s}$ in $H^*(\mathrm{Gr}(b,n))$ is the number of filtered tableaux with shape $\square_b$ whose content is the sequence $((b,\lambda_1),\ldots,(b,\lambda_s))$. We generalize this to any flag manifold.

Theorem 1.2. *Let $\Lambda = ((a_1,\lambda_1),\ldots,(a_s,\lambda_s))$ be a Grassmannian Schubert problem on $\mathcal{F}\ell_\alpha$. Then the coefficient of $[\mathrm{pt}]_\alpha$ in the product $\prod_i \pi^*_{a_i}\sigma_{\lambda_i}$ is the number of filtered tableaux with shape ∇_α and content Λ.*

Example 1.3. We use this formula to compute the intersection number N, defined by

$$N[\mathrm{pt}]_\alpha \;=\; \pi_1^*(\sigma_{\square\square\square})\cdot\pi_2^*(\sigma_{\square\square\square})\cdot\pi_3^*(\sigma_{\boxplus})\cdot\pi_4^*(\sigma_{\substack{\square\\\square\\\square}})\cdot\pi_5^*(\sigma_{\substack{\square\\\square\\\square}})\,.$$

Here, $\alpha = [4]$ and ∇_α is the full staircase shape. There are exactly three sequences of shapes $\mu_\bullet$ which satisfy the condition (1) in the definition of filtered tableaux.



Each of the first two sequences support a unique filtered tableau satisfying condition (2), while the third supports two; thus the required intersection number is 4, which may be verified by direct computation using the Pieri formula for flag manifolds [Sot96]. Indeed, there is a unique Littlewood-Richardson tableau of shape ν/μ and content λ when λ is a single row or column and also when the shapes of ν/μ and λ are the same or rotated by 180°. The only skew shape here which admits more than one Littlewood-Richardson tableau is when $\lambda = \boxplus$ and $\nu/\mu =$ ⊞̸. There are two such Littlewood-Richardson tableaux, given in (1.1), and this occurs in the middle of the third chain.

2. Proof of Theorem 1.2

Let $\mathcal{F}\ell := \mathcal{F}\ell_{[n-1]}$ be the manifold of complete flags in $\mathbb{C}^n$, which has dimension $\binom{n}{2}$. Its Schubert classes are indexed by all permutations w of the numbers $\{1,2,\ldots,n\}$. We prove a strengthening of Theorem 1.2 for the full flag manifold and use this to deduce Theorem 1.2 for all partial flag manifolds. We give the key definition of this section.

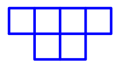
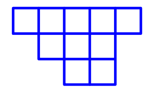
Definition 2.1. A permutation w is a *valley permutation* with *floor at a* if

$$w(1) > w(2) > \cdots > w(a) \qquad\text{and}\qquad w(a{+}1) < w(a{+}2) < \cdots < w(n)\,.$$

For example, 531246 and 643125 are valley permutations with floor at 3. We associate a shape $\mu = \mu(w)$ to any valley permutation w. If w has floor at a, then $\mu(w)$ is the shape whose rows are

$$w(1) - 1 \;>\; w(2) - 1 \;>\; \cdots \;>\; w(a) - 1 \;\geq\; 0\,.$$

This has either a or $a-1$ rows. Observe that w is determined by $\mu(w)$ and that $\ell(w) = |\mu(w)|$ where $\ell(w)$ counts the inversions in w. For example,

$\mu(531246) \;=$  and $\mu(643125) \;=$  .

Theorem 2.2. *Let $\Lambda = ((a_1, \lambda_1), \ldots, (a_t, \lambda_t))$ with $a_1 \leq a_2 \leq \cdots \leq a_t$ and suppose that w is a valley permutation with shape μ. Then the coefficient of σ_w in the product $\prod_{i=1}^t \pi_{a_i}^* \sigma_{\lambda_i}$ in the cohomology ring of $\mathcal{F}\ell$ is the number of filtered tableau with shape μ and content Λ.*

Since the class [pt] of a point in $H^*(\mathcal{F}\ell)$ is indexed by the longest permutation, which is a valley permutation with shape $\nabla_{[n-1]}$, Theorem 2.2 implies Theorem 1.2 for $\mathcal{F}\ell_{[n-1]}$. We deduce Theorem 1.2 for general flag manifolds $\mathcal{F}\ell_\alpha$ from the case for $\mathcal{F}\ell_{[n-1]}$.

Proof of Theorem 1.2. Suppose that $b \notin \alpha$, say $\alpha_i < b < \alpha_{i+1}$, and set $\alpha' := \alpha \cup \{b\}$. We assume that the theorem holds for $\mathcal{F}\ell_{\alpha'}$, and deduce it for $\mathcal{F}\ell_\alpha$.

Let κ be the rectangular partition with $b-\alpha_i$ rows and $\alpha_{i+1}-b$ columns. Set $\Lambda' := ((a_1, \lambda_1), \ldots, (b, \kappa), \ldots, (a_s, \lambda_s))$. Note that $\pi^*_{\alpha',b}\sigma_\kappa$ is dual to $\pi^*_{\alpha',\alpha}[\mathrm{pt}]_\alpha$ in $H^*(\mathcal{F}\ell_{\alpha'})$ under the Poincaré pairing. Thus, for any $\tau \in H^*(\mathcal{F}\ell_\alpha)$ we have

$$\big[[\mathrm{pt}]_{\alpha'}\big]\, \pi^*_{\alpha',b}\sigma_\kappa \cdot \pi^*_{\alpha',\alpha}\tau \;=\; \big[[\mathrm{pt}]_\alpha\big]\,\tau\,,$$

where $\big[[\mathrm{pt}]_\alpha\big]\tau$ denotes the coefficient of $[\mathrm{pt}]_\alpha$ in τ. In particular,

$$\big[[\mathrm{pt}]_{\alpha'}\big] \prod_{(a,\lambda)\in\Lambda} \pi_a^*\sigma_\lambda \;=\; \big[[\mathrm{pt}]_\alpha\big] \prod_{(a',\lambda')\in\Lambda'} \pi_{a'}^*\sigma_{\lambda'}\,. \tag{2.1}$$

There is a bijection between filtered tableaux with shape ∇_α and content Λ and those with shape $\nabla_{\alpha'}$ and content Λ', obtained by inserting the unique Littlewood-Richardson tableau of shape and content κ into the filtration. Thus counting either set of filtered tableaux gives the coefficient (2.1).

A Schubert class σ_w *appears* in a product $\sigma_u \cdots \sigma_v$ of Schubert classes if, when we expand the product in the basis of Schubert classes, σ_w appears with a positive coefficient.

We will prove Theorem 2.2 by induction on the number of terms t in the product. Important for this is the following proposition which summarizes some discussion at the beginning of Section 1 in [BS98].

Proposition 2.3. *If a Schubert class σ_w appears in the product $\sigma_v \cdot \pi_a^*\sigma_\lambda$, then the following conditions hold.*

1. *Whenever $i \leq a < j$, we have $w(i) \geq v(i)$ and $w(j) \leq v(j)$.*
2. *If $i < j \leq a$ and $v(i) < v(j)$, then $w(i) < w(j)$. If $a < i < j$ and $v(i) < v(j)$, then $w(i) < w(j)$.*

In [BS98], it is shown that the conditions in Proposition 2.3 define an order relation $v \leq_a w$, which is a suborder of the Bruhat order. We deduce an important lemma.

Lemma 2.4. *If σ_w appears in $\prod_{i=1}^{t} \pi_{a_i}^* \sigma_{\lambda_i}$ then w has no descents after position a_t.*

Proof. We prove this by induction on t. It holds when $t = 0$, as the multiplicative identity in cohomology is the Schubert class indexed by the identity permutation.

Suppose that σ_w appears in the product $\prod_{i=1}^{t} \pi_{a_i}^* \sigma_{\lambda_i}$. Then there is some permutation v such that σ_v appears in the product $\prod_{i=1}^{t-1} \pi_{a_i}^* \sigma_{\lambda_i}$ and σ_w appears in the product $\sigma_v \cdot \pi_{a_t}^* \sigma_{\lambda_t}$. Hence $v \leq_{a_t} w$. Since v has no descents after position a_{t-1} and $a_{t-1} \leq a_t$, condition (2) of Proposition 2.3 implies that w has no descents after position a_t. $\square$

For permutations v, w and a partition $\lambda \subset \square_a$, let $c^w_{v,a,\lambda}$ be the coefficient of σ_w in the product $\sigma_v \cdot \pi_a^* \sigma_\lambda$. One of the main results in [BS98] is the following identity.

Proposition 2.5. *Suppose that $v \leq_a w$ and $x \leq_a z$ with $wv^{-1} = zx^{-1}$. Then for every $\lambda \subset \square_a$ we have $c^w_{v,a,\lambda} = c^z_{x,a,\lambda}$.*

Suppose that a shape $\nu \subset \nabla_{[n-1]}$ has either $b-1$ or b rows. We define $\nu|_b$ to be the intersection of the shape ν with $\square_b$.

Proof of Theorem 2.2. We proceed by induction on t. The theorem holds (trivially) for $t = 0$; assume that $t > 0$ and that it holds for $t - 1$.

Let w be a valley permutation with shape μ, and suppose that w appears in the product $\prod_{i=1}^{t} \pi_{a_i}^* \sigma_{\lambda_i}$. Then by Lemma 2.4, w has a floor at a_t. Let us expand the product

$$\prod_{i=1}^{t-1} \pi_{a_i}^* \sigma_{\lambda_i} \;=\; \sum_{v} c^v \sigma_v \,.$$

Then the coefficient of σ_w in the product $\prod_{i=1}^{t} \pi_{a_i}^* \sigma_{\lambda_i}$ is the sum

$$\sum_{v \leq_{a_t} w} c^v \cdot c^w_{v,a_t,\lambda_t} \,.$$

Suppose that $v \leq_{a_t} w$. Since w has a floor at a_t, Proposition 2.3(2) implies that

$$v(1) \;>\; v(2) \;>\; \cdots \;>\; v(a_t) \,.$$

If the coefficient $c^v \neq 0$, so that v can contribute to this sum, then Lemma 2.4 implies that v has no descents after position a_{t-1}. Since $a_t - 1 \leq a_{t-1} \leq a_t$, this implies that v is a valley permutation with a floor at a_t.

Let ν be the shape of v. Since both w and v have floor at a_t, both μ and ν have either $a_t - 1$ or a_t rows, and thus $\mu/\nu \subset \square_{a_t}$. The theorem would follow if we knew that

$$c^w_{v,a_t,\lambda_t} \;=\; c^{\mu/\nu}_{\lambda_t} \,. \tag{2.2}$$

To see this, note that there is a bijection between filtered tableaux on μ with content $((a_1, \lambda_1), \ldots, (a_t, \lambda_t))$ and triples $(\nu, T_\bullet, T)$ where $\nu \subset \mu$, $T_\bullet$ is a filtered tableau of shape ν and content $((a_1, \lambda_1), \ldots, (a_{t-1}, \lambda_{t-1}))$, and T is a Littlewood-Richardson tableau of shape μ/ν and content λ; hence the number of these is

$$\sum_{v \leq_{a_t} w} c^v \cdot c^{\mu/\nu}_{\lambda_t} \,.$$

But (2.2) follows from Proposition 2.5. Let x (respectively z) be the permutation obtained from v (respectively from w) by reversing the first a_t values, i.e.

$$x(i) \;=\; \begin{cases} v(a_t+1-i) & \text{if } 1 \le i \le a_t \\ v(i) & \text{otherwise.} \end{cases}$$

Then x and z are Grassmannian permutations with descent a_t, and shapes $\nu|_{a_t}$ and $\mu|_{a_t}$, respectively, and $\mu/\nu = (\mu|_{a_t})/(\nu|_{a_t})$. Furthermore, $x \le_{a_t} z$ and $wv^{-1} = zx^{-1}$, from which we deduce (2.2).

3. Further Remarks

When all the classes σ_{λ_i} have degree 2 ($\lambda_i = \square$, a single box), the multiplication formula $\sigma_w \cdot \pi^*_{a_i}\square$ is due to Monk [Mon59]. Monk's formula states that

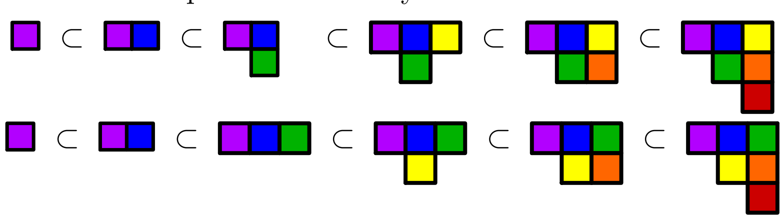
$$\sigma_w \cdot \pi^*_{a_i}\square \;=\; \sum_{\substack{j \le i < k \\ \ell(wr_{jk}) = \ell(w)+1}} \sigma_{wr_{jk}}\,, \tag{3.1}$$

where $r_{jk} \in S_n$ is the transposition swapping j and k. Iterating Monk's formula one sees that the coefficient of $[\mathrm{pt}]_\alpha$ in a product $\prod_{i=1}^{\dim(\alpha)} \pi^*_{a_i}\square$ is obtained by counting certain chains in the Bruhat order. It is not hard to see directly from (3.1) that each permutation w in such a chain corresponds to a shape μ in $\triangledown_\alpha$ such that the number of boxes in the column j of μ equals $\#\{k \in [j] \mid w(k) > w(j+1)\}$, for all $j \in \{\min(\alpha), \ldots, n-1\}$. Indeed, if the permutation w does not correspond to a shape, then no term on the right hand side of (3.1) corresponds to a shape. It follows that the coefficient is the number of chains of shapes in $\triangledown_\alpha$ where the ith step involves adding a box in $\square_{a_i}$, which is the answer given by our formula.

For example, we have

$$2[\mathrm{pt}]_{[3]} \;=\; \pi^*_1\square \cdot \pi^*_1\square \cdot \pi^*_2\square \cdot \pi^*_2\square \cdot \pi^*_3\square \cdot \pi^*_3\square\,,$$

as there are two chains of shapes which satisfy this condition.



It is possible to give a purely combinatorial proof of Theorem 1.2 using Kogan's formula [Kog01, Theorem 2.4]. This rule is based on insertion of RC-graphs and gives the coefficient $c^w_{v,a,\lambda}$, when $v(a{+}1) < v(a{+}2) < \cdots < v(n)$. In particular, this gives a formula for the product when v and w are a valley permutations with a floor at a, and so we may use this in a formula for the intersection numbers of Theorem 1.2 to give a combinatorial proof.

The conventions in [Kog01] for Schubert classes differ from those used in this article. To compare conventions, it is necessary to replace our permutations w by $\widetilde{w} = w_0ww_0$ throughout. In particular, a cohomology class indexed by w in this article is the class

indexed by $\widetilde{w}$ in [Kog01]. Thus our condition on v becomes $\widetilde{v}(1) < \widetilde{v}(2) < \cdots < \widetilde{v}(a)$, which is the condition found in [Kog01].

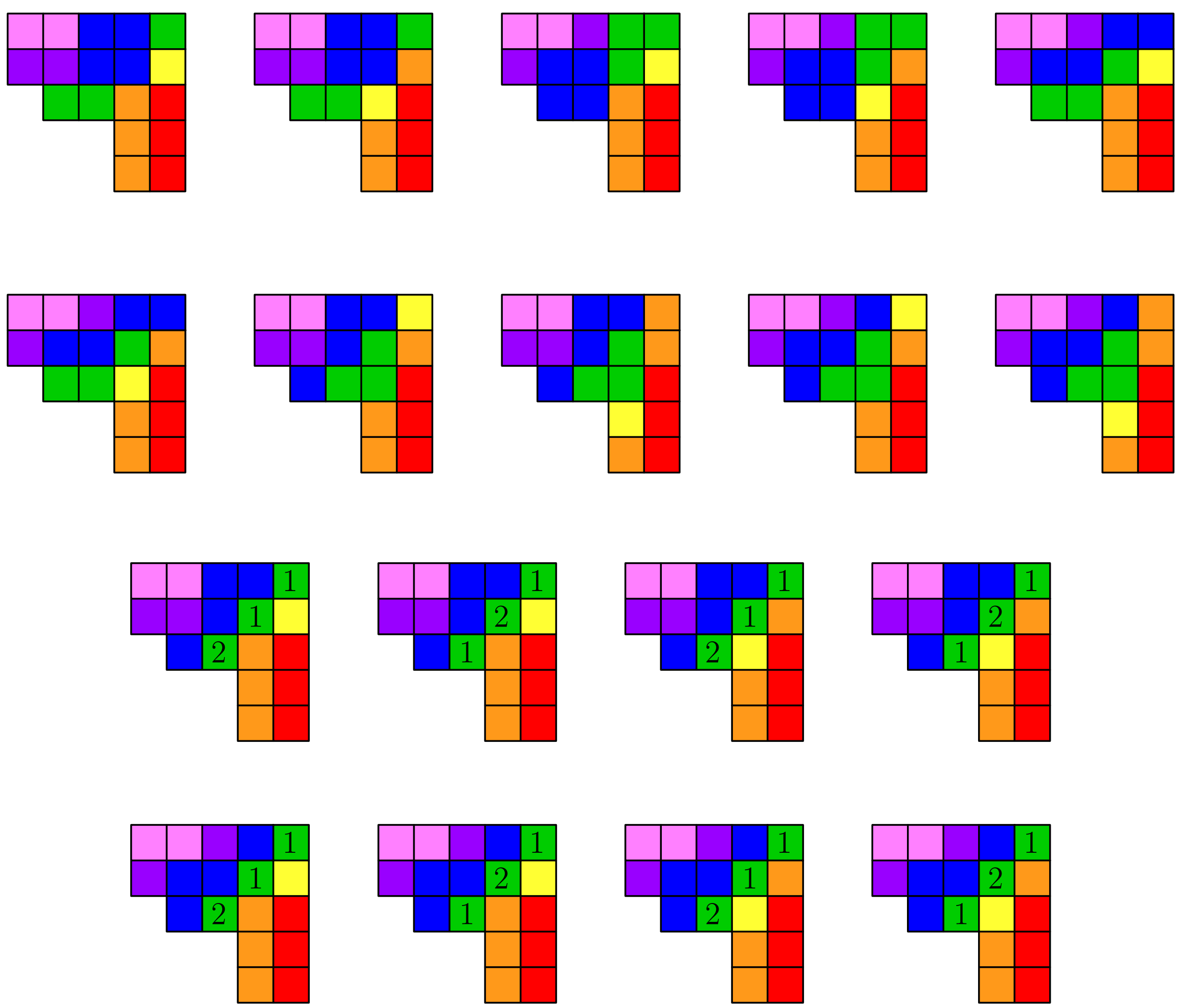
To deduce Theorem 1.2 from this formula, we would need to show that, for valley permutations w, v with floor at a, Kogan's rule for $c^w_{v,a,\lambda}$ coincides with the Littlewood-Richardson rule for $c^{\mu/\nu}_\lambda$, where $\nu = \mu(v)|_a$ and $\mu = \mu(w)|_a$. Here, $\mu(v)$ is the shape of v and $\mu(w)$ is the shape of w. While this is certainly possible, we chose not to pursue this.

Appendix A. More examples

Example A.1. Consider the following product in $\mathcal{F}\ell_{235}$,

$$\pi_2^*(\sigma_{\square\square}) \cdot \pi_2^*(\sigma_{\square\square}) \cdot \pi_3^*(\sigma_{\boxplus}) \cdot \pi_3^*(\sigma_{\substack{\square\square\\ \square\ \ }}) \cdot \pi_5^*(\sigma_{\square}) \cdot \pi_5^*(\sigma_{\substack{\square\\ \square\\ \square}}) \cdot \pi_5^*(\sigma_{\substack{\square\\ \square\\ \square}})\,.$$

By Theorem 1.2, the coefficient of [pt] is the number of filtered tableau with content $((2, \square\square), (2, \square\square), (3, \boxplus), (3, \substack{\square\square\\ \square\ \ }), (5, \square), (5, \substack{\square\\ \square\\ \square}), (5, \substack{\square\\ \square\\ \square}))$, which is 18:



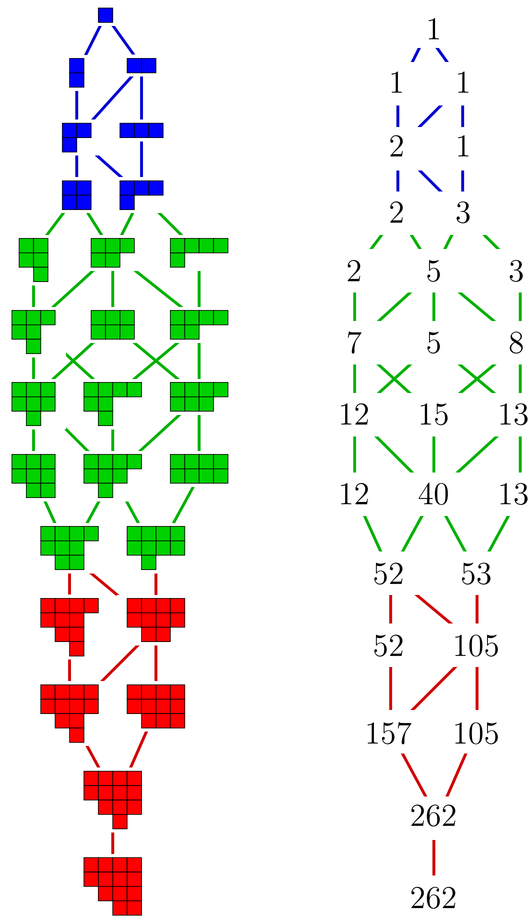
Example A.2. We remarked in Section 3 that, when every partition is a single box ($\lambda_i = \square$), a filtered tableau is a particular saturated chain of shapes in ∇_α. When $n = 6$ we look at this for the problem

$$(\pi_2^*\square)^4 \cdot (\pi_3^*\square)^5 \cdot (\pi_4^*\square)^4$$

in $\mathcal{F}\ell_{234}$.

To the right is the poset of shapes μ in ∇_{234}, where at level t (from the top) the shape has at most a_t and at least $a_t - 1$ rows.

Further to the right, we count the number of chains in this poset, which shows that the intersection number is 262.



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